

Relativistic Shifted - l Expansion Technique for Dirac and Klein-Gordon Equations

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Abstract

The shifted - l expansion technique (SLET) is extended to solve for Dirac particle trapped in spherically symmetric scalar and/or 4-vector potentials. A parameter $\lambda = 0, 1$ is introduced in such a way that one can obtain the Klein-Gordon (KG) bound states from Dirac bound states. The 4-vector Coulomb, the scalar linear, and the equally mixed scalar and 4-vector power-law potentials are used in KG and Dirac equations. Exact numerical results are obtained for the 4-vector Coulomb potential in both KG and Dirac equations. Highly accurate and fast converging results are obtained for the scalar linear and the equally mixed scalar and 4-vector power-law potentials.

I Introduction

In a previous paper we have developed the shifted- l expansion technique (SLET) to solve 3D and 2D Schrödinger equations [1]. Therein, we have suggested SLET as a reformation to the shifted large-N expansion technique (SLNT) [2,3]. The importance of the Klein-Gordon (KG) and Dirac wave equations with scalar and/or 4-vector potentials arises in many fields of physics [4-24]. In the framework of the mentioned equations, a scalar or an equally mixed scalar and 4-vector potentials have considerable interest in the study of quarkonium systems [6,8-10,12,16-19]. The mixture of 4-vector and scalar potentials has been studied by Long and Robson [20]. The 4-vector potentials have great utility in atomic, nuclear, and plasma physics [21-24]. Therefore, many attempts have been made to develop approximation techniques to treat relativistic particles in the KG and Dirac wave equations [4-24].

To the best of our knowledge, the attempts that have been made to develop a relativistic $1/N$ expansion technique include, Neito [14] who has extended the unshifted $1/N$ expansion [25] formalism to treat a π -mesonic atom in the KG equation. Miramontes and Pajares [15] who have studied the same technique in the framework of the KG and Dirac equations. Iterative procedures concerning the $1/N$ expansion technique were introduced to deal with KG and Dirac particles [11]. Stepanove and Tutik [10] who have used the \hbar -expansion procedure in a similar manner to the $1/N$ expansion procedure. Such procedures are straight forward though rather tedious and suffer from convergence problems in comparison to SLNT [4-8]. In addition, highly excited states pose problems for the above mentioned techniques. Recently, Roychoudhury and Varshni have developed SLNT to treat scalar [12] and 4-vector potentials [13] in the Dirac equation. Panja et al [4] have applied SLNT to Dirac particle by considering the KG equation in which a spin-orbit interaction is introduced analogous to Pauli theory (this has been proposed by Papp [8]). Mustafa and Sever [6-8] have used a different approach to SLNT for the KG and Dirac equations.

Encouraged by the success of SLET in the Schrödinger equation and the importance of KG and Dirac equations with scalar and/or 4-vector potentials, we feel tempted to extend SLET to solve for the bound states of the above equations. We shall do so by considering the Dirac equation in which we introduce a parameter $\lambda = 0, 1$ in the spin-dependent term. For $\lambda = 0$,

Dirac equation implies KG equation with scalar and/or 4-vector potentials. In this case, the total angular momentum j is given through the relation $l = j \pm 1/2$, and the radial wave function of KG particle is the radial large-component of the Dirac spinor. When $\lambda = 1$, the following are the bound states under consideration; (i) the Dirac bound states in scalar and/or 4-vector potentials, and (ii) the Dirac bound states in an equally mixed scalar and 4-vector potentials. The bound states in (ii) are identical to KG bound states in the same structure potentials.

For the sake of numerical illustration on the accuracy of our SLET we discuss the results for the following particles; (i) Dirac and KG particles in 4-vector Coulomb potentials, (ii) Dirac and KG particles in scalar linear potentials, and (iii) Dirac particle in an equally mixed scalar and 4-vector power-law potential. Elsewhere, we shall investigate Dirac and KG particles in a mixture of 4-vector and scalar potentials.

In Sec II we shall extend SLET to solve for Dirac and KG bound states. Analytical expressions are given in such away that allows the reader to use them without proceeding into their derivations. In Sec III we shall show that the analytical expressions of Sec II lead to highly accurate and fast converging numerical results when applied to the 4-vector Coulomb, the linear scalar, and the equally mixed scalar and 4-vector power-law potentials in both Dirac and KG equations. We conclude and remark in Sec IV.

II SLET for Dirac and KG bound states.

With a scalar $S(r)$ and a 4-vector $V(r)$ Lorentz interactions, simultaneously present, the radial coupled Dirac equations can be written [16,26] ($\hbar = c = 1$) as

$$\xi_1 G(r) + \frac{dF(r)}{dr} - \frac{k}{r} F(r) = 0, \quad (1)$$

$$\xi_2 F(r) - \frac{dG(r)}{dr} - \frac{k}{r} G(r) = 0. \quad (2)$$

Where $\xi_1 = E - m - V(r) - S(r)$, $\xi_2 = E + m - y(r)$, $y(r) = V(r) - S(r)$, and E is the relativistic energy. $G(r)$ and $F(r)$ are the large and small radial components of the Dirac spinor [19], respectively. Provided $V(r)$ and $S(r)$ are spherically symmetric, $k = -(l + 1)$ for the total angular momentum

$j = l + 1/2$, $k = l$ for $j = l - 1/2$, and l is angular momentum quantum number.

In terms of the large-component $G(r)$ of the Dirac spinor, Eq.(1) reads

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + \frac{1}{\xi_2} (y'(r) \left[\frac{d}{dr} + \frac{k}{r} \right]) + \xi_1 \xi_2 \right] G(r) = 0, \quad (3)$$

where the prime denotes d/dr .

For any k one can show that $k(k+1) = l(l+1)$. Furthermore, we remove the first derivative by proposing the ansatz

$$G(r) = R(r) \exp(-p(r)/2) \quad ; \quad p'(r) = y'(r)/\xi_2. \quad (4)$$

Which in turn implies the Schrödinger-like radial Dirac equation

$$\left[-d^2/dr^2 + l(l+1)/r^2 + U(r) - \xi_1 \xi_2 \right] R(r) = 0, \quad (5)$$

with

$$U(r) = \frac{1}{4} \left[\frac{2y''(r)}{\xi_2} - \frac{4kp'(r)}{r} + 3p'(r)^2 \right]. \quad (6)$$

It should be pointed out that the choice of the ansatz in Eq.(4) is motivated by requiring the agreement between Eq.(5) and the outcome of the linear matrix eigenvalue problem, Eq.(1) and (2), after being transformed into a diagonal Sturm-Liouville eigenvalue system. For more details on this system the reader may refer to Barut [27]. Moreover, it is evident that Eq.(5) reduces to KG wave equation with scalar and/or 4-vector potentials provided $U(r) = 0$ [6]. We therefore introduce a parameter $\lambda = 0, 1$ in $U(r)$ so that $\lambda = 0$ and $\lambda = 1$ correspond to KG and Dirac wave equations, respectively. Also, we shall be interested in problems where the rest energy is large compared to the binding energy $W = E - m$. Eq.(6) thus becomes

$$U(r) = \frac{\lambda}{4m} \left[y''(r) - \frac{2ky'(r)}{r} + \frac{3y'(r)^2}{4m} \right], \quad (7)$$

where we have considered

$$\frac{1}{\xi_2} = \frac{1}{W + 2m - y(r)} \simeq \frac{1}{2m} \left(1 - \frac{W - y(r)}{2m} + \dots \right), \quad (8)$$

and the term $(W - y(r))/4m^2$ is small enough to be neglected.

If we shift the angular momentum l through the relation $l = \bar{l} + \beta$ and define

$$\gamma(r) = -V(r)^2 + m(r)^2 + U(r), \quad (9)$$

where

$$m(r) = m + S(r), \quad (10)$$

Eq.(5) becomes

$$\left[-\frac{d^2}{dr^2} + \frac{[\bar{l}^2 + \bar{l}(2\beta + 1) + \beta(\beta + 1)]}{r^2} + \gamma(r) + 2EV(r) \right] R(r) = E^2 R(r). \quad (11)$$

We shall now start the systematic $1/\bar{l}$ expansion by defining

$$\gamma(r) = \frac{\bar{l}^2}{Q} \left[\gamma(r_o) + \gamma'(r_o)r_o x/\bar{l}^{1/2} + \gamma''(r_o)r_o^2 x^2/2\bar{l} + \dots \right], \quad (12)$$

$$V(r) = \frac{\bar{l}^2}{Q} \left[V(r_o) + V'(r_o)r_o x/\bar{l}^{1/2} + V''(r_o)r_o^2 x^2/2\bar{l} + \dots \right], \quad (13)$$

$$E = \frac{\bar{l}^2}{Q} \left[E_o + E_1/\bar{l} + E_2/\bar{l}^2 + E_3/\bar{l}^3 + \dots \right]. \quad (14)$$

where $x = \bar{l}^{1/2}(r - r_o)/r_o$, and Q is to be set equal to \bar{l}^2 at the end of the calculations. With Eqs.(12)-(14) into Eq.(11) one gets

$$\begin{aligned} & \left[\frac{-d^2}{dx^2} + (\bar{l} + (2\beta + 1) + \frac{\beta(\beta + 1)}{\bar{l}})(1 - \frac{2x}{\bar{l}^{1/2}} + \frac{3x^2}{\bar{l}} - \dots) \right. \\ & + \frac{r_o^2 \bar{l}}{Q} (\gamma(r_o) + \frac{\gamma'(r_o)r_o x}{\bar{l}^{1/2}} + \frac{\gamma''(r_o)r_o^2 x^2}{2\bar{l}} + \frac{\gamma'''(r_o)r_o^3 x^3}{6\bar{l}^{3/2}} + \dots) \\ & + \frac{2r_o^2 \bar{l}}{Q} (V(r_o) + \frac{V'(r_o)r_o x}{\bar{l}^{1/2}} + \dots)(E_o + \frac{E_1}{\bar{l}} + \frac{E_2}{\bar{l}^2} + \dots) \left. \right] \phi_{n_r}(x) \\ & = \mu_{n_r} \phi_{n_r}(x) \end{aligned} \quad (15)$$

where

$$\mu_{n_r} = \frac{r_o^2 \bar{l}}{Q} \left[E_o^2 + \frac{2E_o E_1}{\bar{l}} + \frac{(E_1^2 + 2E_o E_2)}{\bar{l}^2} + \frac{2(E_o E_3 + E_1 E_2)}{\bar{l}^3} + \dots \right]. \quad (16)$$

Eq(15) is a Schrödinger-like equation for the one-dimensional anharmonic oscillator and has been discussed in detail by Imbo et al [2]. We therefore quote only the resulting eigenvalue of Ref.[2] and write

$$\begin{aligned} \mu_{n_r} = & \bar{l} \left[1 + \frac{2r_o^2 V(r_o) E_o}{Q} + \frac{r_o^2 \gamma(r_o)}{Q} \right] \\ & + \left[(2\beta + 1) + \frac{2r_o^2 V(r_o) E_1}{Q} + (n_r + \frac{1}{2})w \right] \\ & + \frac{1}{\bar{l}} \left[\beta(\beta + 1) + \frac{2r_o^2 V(r_o) E_2}{Q} + \alpha_1 \right] \\ & + \frac{1}{\bar{l}^2} \left[\frac{2r_o^2 V(r_o) E_3}{Q} + \alpha_2 \right], \end{aligned} \quad (17)$$

where α_1 and α_2 are given in the appendix of Ref. [1], and the corresponding relativistic ϵ 's and δ 's are given in the appendix of this paper. If we now compare Eq.(17) with (16), we obtain

$$E_o = V(r_o) + \sqrt{V(r_o)^2 + Q/r_o^2 + \gamma(r_o)}, \quad (18)$$

$$E_1 = \frac{Q}{2r_o^2(E_o - V(r_o))} [2\beta + 1 + (n_r + 1/2)w], \quad (19)$$

$$E_2 = \frac{Q}{2r_o^2(E_o - V(r_o))} [\beta(\beta + 1) + \alpha_1], \quad (20)$$

$$E_3 = \frac{Q}{2r_o^2(E_o - V(r_o))} \alpha_2, \quad (21)$$

and

$$E_{n_r} = E_o + \frac{1}{2r_o^2(E_o - V(r_o))} \left[\beta(\beta + 1) + \alpha_1 + \frac{\alpha_2}{\bar{l}} \right]. \quad (22)$$

r_o is chosen to be the minimum of E_o , i. e.;

$$dE_o/dr_o = 0 \quad \text{and} \quad d^2E_o/dr_o^2 > 0. \quad (23)$$

Hence, r_o is obtained through the relation

$$2(l - \beta)^2 = b(r_o) + \sqrt{b(r_o)^2 - 4c(r_o)}, \quad (24)$$

Where

$$b(r_o) = r_o^3 [2V(r_o)V'(r_o) + \gamma'(r_o) + r_o V'(r_o)^2], \quad (25)$$

$$c(r_o) = \frac{r_o^6}{4} [\gamma'(r_o)^2 + 4V(r_o)V'(r_o)\gamma'(r_o) - 4\gamma(r_o)V'(r_o)^2] \quad (26)$$

The shifting parameter β is determined by requiring $E_1 = 0$ [1,2]. Therefore,

$$\beta = -[1 + (n_r + 1/2)w]/2, \quad (27)$$

where

$$w = \left[12 + \frac{2r_o^4 \gamma''(r_o)}{Q} + \frac{4r_o^4 V''(r_o) E_o}{Q} \right]^{1/2}. \quad (28)$$

III Applications, numerical results, and discussion.

In this section we shall consider the 4-vector Coulomb, the scalar linear, and the equally-mixed scalar and 4-vector power-law potentials in KG and Dirac equations. To obtain the KG and Dirac bound states we set $\lambda = 0$ and $\lambda = 1$, respectively. r_o is obtained from Eq.(24) through (25)-(29), and the energy eigenvalues are calculated from Eq.(22). We shall examine the accuracy and the convergence of SLET by comparing its outcomes with the exact and numerical results of other methods.

III.1 The 4-vector Coulomb potential in KG and Dirac equations [$V(r) = -\alpha/r$, $S(r) = 0$]

A pionic atom in a Coulomb-type 4-vector potential, $V(r) = -\alpha/r$ and $S(r) = 0$, obeys the KG equation. Where α is the fine structure constant and $\alpha = 1/137.03602$. The pion mass is $m_\pi c^2 = 139.577$ MeV and spin-0. An electron trapped in the same type of potential obeys the Dirac equation. The rest mass of the electron is $m_e c^2 = 0.5110041$ MeV. However, for both particles exact analytical results for binding energies exist [26] and hence exact numerical results can be reproduced and compared with those of SLET.

In Tables 1 and 2 we show the results of SLET compared with the exact results for KG and Dirac particles, respectively. The results are shown in such a way that one can easily judge the accuracy and the convergence. Our results appear in excellent agreement with the exact ones. The minimum and maximum percent accuracies found for both particles are 99.998% and 100.000%, respectively. The convergence is noted to be very fast in the sense that the leading term E_o contributes from 99.996% to 100.000% of the exact binding energy.

III.2 The scalar linear potential in KG and Dirac equations [$V(r) = 0$, and $S(r) = Ar$]

A pure scalar linear potential, $V(r) = 0$ and $S(r) = Ar$, in KG and Dirac equations is precisely a quark confining linear potential. This potential has been studied by Gunion and Li [17] in KG and Dirac equations and hence accurate numerical results exist for comparison with SLET results. However, Mustafa et al [6] and Roy et al [12] have used SLNT to calculate part of the KG and Dirac mass spectra, respectively, of some quark- antiquark systems.

In Tables 3-5 we compare SLET results with those of SLNT [6,12] and those of Gunion and Li [17]. Comparing our results, in Table 3, with the numerically predicted ones [17] we have scored a minimum accuracy of 99.96% and a maximum accuracy of 100.00%. In Table 4 the minimum accuracy is noted to be 99.61% and the maximum is 99.76%. In Table 5 the accuracy ranges from 99.05% to 100.00%. Furthermore, the energy eigenvalues given by Eq.(14) have been noted to be fast converging in the sense that the leading term E_o contributes more than 99% of the total energy. While our results in Table 3 are in exact agreement with those of SLNT [6], they are only in qual-

itative agreement with those of SLNT [12] in Tables 4 and 5. The term that appears as $3y'(r)^2/8m^2$ in Eq. (7) of this text has appeared as $y'(r)^2/8m^2$ in Eq.(10a) of Ref.[12]. Their method [12] was restricted only to the scalar linear potential $S(r) = Ar$ in the Dirac equation where the effective potential can easily be treated by the usual nonrelativistic SLNT. In addition the authors of Ref.[12] were interested in terms only up to $O(A/m^2)$, a thing we have not assumed here. However, the term $3y'(r)^2/8m^2$ has been confirmed to exist by Barut [27]. Moreover, it is noteworthy to mention that for some values of n_r and l the standard numerical method [17] could not predict the effect of the spin-orbit coupling term $\vec{L} \cdot \vec{S}$. SLET was sensitive enough to predict this effect. To see this one may compare the results of Table 3 with those of Table 4.

III.3 The equally mixed scalar and 4-vector power-law potential $[V(r) = S(r) = Ar^\nu + V_o]$

The power-law potential of the form

$$V(r) = Ar^\nu + V_o \quad (29)$$

with $\nu = 0.1$ and $A > 0$ is a simple non-QCD- based potential. An equally mixed scalar and 4-vector structure of this potential, i.e. $V(r) = S(r)$, in Dirac equation is known to reproduce the data of Ψ and Υ spectroscopies.

In Tables 6,7,8 and 9 we show the spin-averaged results for the equally mixed scalar and 4-vector power-law potential, Eq.(29), with $V_o = -2.028$ GeV, $\nu = 0.1$, $A = a^{\nu+1}$, $a=1.709$, $m_c = 1.6179$ GeV, $m_b = 5.0114$ GeV, $m_s = 0.325$ GeV, and $m_d = 0.01$ GeV.

In Table 6 we compare the eigenvalues $\varepsilon_{n_r l}$ corresponding to the confined bound states of quarks. $\varepsilon_{n_r l}$ is calculated through the relation [19]

$$\varepsilon_{n_r l} = (E_c - m_c - 2V_o) \left[(E_c + m_c)(2A)^{-2/\nu} \right]^{\nu/(\nu+2)} \quad (30)$$

In this table the accuracies of our results are in the range 99.90% to 100.00%. Whereas in Ref.[5], SLNT has scored a maximum error of 2.7% corresponding to a minimum accuracy of 97.3%. Therein, the authors [5] should have noticed that the term on the right hand side of Eq.(5), in Ref.[5], vanishes for the equally mixed scalar and 4-vector potential case. This makes it unnecessary to seek the nonrelativistic limit they have sought. The consideration of

this limit in Ref.[5] has caused a spin- orbit coupling term to exist (see Eq.(9) of Ref. [5]) although it is well known that one of the interesting features of such mixture cancels any spin-orbit coupling effect [18]. In Tables 7, 8 and 9 we investigate the accuracy of SLET for some heavy, $b\bar{b}$, light, $s\bar{s}$, and non-self-conjugate atom like $c\bar{d}$ mesons, respectively. Therein, SLET results are found comparable to the numerically predicted ones [19].

IV Conclusions and remarks

In this paper we have extended the shifted- l expansion technique (SLET) to solve for the eigenvalues of KG and Dirac equations with scalar and/or 4-vector potentials. We have shown that Dirac equation reduces to KG equation, provided that the angular momentum quantum number is given through the relation $l = j \pm 1/2$, and the radial KG wave function is the radial large-component of the Dirac spinor. A parameter $\lambda = 0, 1$ has been introduced in the spin-dependent term $U(r)$ so that when $\lambda = 0$ the Dirac equation reduces to the KG equation. The formalism developed in Sec II has been investigated through the 4-vector Coulomb, the scalar linear, and the equally mixed scalar and 4-vector potentials in Dirac and KG equations. SLET results were found in excellent agreements with the exact and numerically predicted results found elsewhere.

To the best of our knowledge, this is the first time that both Dirac and KG equations, with arbitrary scalar and/or 4-vector potentials, have been solved through one analytical procedure that yields highly accurate and fast converging results.

In conclusion, SLET is to be understood as being an expansion through some existing quantum number, which depends on the symmetry of the potential of interest, in any Schrödinger - like eigenvalue problem. For example, in the case of spherically symmetric potentials the expansion parameter is $1/\bar{l}$ where $\bar{l} = l - \beta$ and in the case of cylindrically symmetric potentials $\bar{l} = |m| - \beta$ where m is the magnetic quantum number. This understanding suggests we should not worry about the N -dimensional form of the wave equation in hand and we should expand directly through the quantum numbers involved in the problem. Therefore, the difficulties associated with inflating the dimensions of the Dirac equation are resolved by using SLET. The attendant method seems more flexible than the SLNT.

Finally, we would like to mention that the formalism we have introduced in this work is also applicable to more complicated relativistic potentials. In particular, screened Coulomb potentials which have great utility in atomic, nuclear, and plasma physics. Also, the wave functions, and normalizations for spin-1/2 and spin-0 particles in spherically symmetric scalar and/or 4-vector potentials can be obtained.

Appendix

The definition for ε'_j s and δ'_i s in α_1 and α_2 of Ref.[1] are as follows

$$\varepsilon_1 = -2(2\beta + 1) \quad , \quad \varepsilon_2 = 3(2\beta + 1), \quad (31)$$

$$\varepsilon_3 = -4 + \frac{r_o^5}{6Q} [\gamma'''(r_o) + 2V'''(r_o)E_o], \quad (32)$$

$$\varepsilon_4 = 5 + \frac{r_o^6}{24Q} [\gamma''''(r_o) + 2V''''(r_o)E_o], \quad (33)$$

$$\delta_1 = -2\beta(\beta + 1) + \frac{2r_o^3 V'(r_o)E_2}{Q} \quad (34)$$

$$\delta_2 = 3\beta(\beta + 1) + \frac{r_o^4 V''(r_o)E_2}{Q}, \quad (35)$$

$$\delta_3 = -4(2\beta + 1) \quad , \quad \delta_4 = 5(2\beta + 1), \quad (36)$$

$$\delta_5 = -6 + \frac{r_o^7}{120Q} [\gamma'''''(r_o) + 2V'''''(r_o)E_o], \quad (37)$$

$$\delta_6 = 7 + \frac{r_o^8}{720Q} [\gamma''''''(r_o) + 2V''''''(r_o)E_o]. \quad (38)$$

The terms including E_1 have been dropped from the expressions above since $E_1=0$.

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Table 1: Comparison of the energy levels (in keV units) for spin-0 particle in $V(r) = -\alpha/r$ and $S(r) = 0$. W represents the binding energies, where $W_o = (E_o - 1)m_\pi c^2$, $W_2 = (E_o + E_2/\bar{l}^2 - 1)m_\pi c^2$, and $W_3 = (E_o + E_2/\bar{l}^2 + E_3/\bar{l}^3 - 1)m_\pi c^2$.

State	$-W_o$	$-W_2$	$-W_3$	Exact[26]
$1s$	3.71649	3.71653	3.71656	3.71659
$2s$	0.92910	0.92910	0.92911	0.92912
$2P$	0.92910	0.92910	0.92910	0.92909
$3s$	0.41293	0.41293	0.41293	0.41294
$3P$	0.41293	0.41293	0.41293	0.41293
$3d$	0.41293	0.41293	0.41293	0.41293
$4s$	0.23227	0.23227	0.23227	0.23228
$4P$	0.23227	0.23227	0.23227	0.23227
$4d$	0.23227	0.23227	0.23227	0.23227
$4f$	0.23227	0.23227	0.23227	0.23227

Table 2: Comparison of the energy levels (in eV units) for spin-1/2 particle in $V(r) = -\alpha/r$ and $S(r) = 0$. W represents the binding energies, where $W_o = (E_o - 1)m_e c^2$, $W_2 = (E_o + E_2/\bar{l}^2 - 1)m_e c^2$, and $W_3 = (E_o + E_2/\bar{l}^2 + E_3/\bar{l}^3 - 1)m_e c^2$.

State	$-W_o$	$-W_2$	$-W_3$	Exact[26]
$1s_{1/2}$	13.60639	13.60657	13.60667	13.60603
$2s_{1/2}$	3.40151	3.40153	3.40155	3.40152
$2P_{1/2}$	3.40150	3.40151	3.40151	3.40152
$2P_{3/2}$	3.40147	3.40147	3.40147	3.40148
$3s_{1/2}$	1.51177	1.51177	1.51178	1.51178
$3P_{1/2}$	1.51177	1.51178	1.51177	1.51178
$3P_{3/2}$	1.51177	1.51177	1.51177	1.51177
$3d_{3/2}$	1.51177	1.51177	1.51177	1.51177
$3d_{5/2}$	1.51176	1.51176	1.51176	1.51177
$4s_{1/2}$	0.85037	0.85037	0.85037	0.85038
$4P_{1/2}$	0.85037	0.85037	0.85037	0.85038
$4P_{3/2}$	0.85037	0.85037	0.85037	0.85037
$4d_{3/2}$	0.85037	0.85037	0.85037	0.85037
$4d_{5/2}$	0.85037	0.85037	0.85037	0.85037
$4f_{5/2}$	0.85037	0.85037	0.85037	0.85037
$4f_{7/2}$	0.85037	0.85037	0.85037	0.85037

Table 3: KG results for part of the mass spectra (in GeV) of Ψ, Ψ' system with $S(r) = Ar$, $A = 0.137\text{GeV}^2$ and $m = 1.12\text{GeV}$. The values in round parentheses are those of SLNT [6]. The values in square parentheses are those of Ref.[17].

n_r	l			
	0	1	2	3
0	3.12	3.46	3.74	3.99
	(3.12)	(3.46)	(3.74)	(3.99)
	[3.1]	[3.47]	[3.73]	[3.98]
1	3.7	3.95	4.18	4.387
	(3.7)	(3.96)	(4.18)	(4.387)
	[3.7]	[3.95]	[4.17]	[4.387]
2	4.15	4.36	4.56	4.74
	(4.16)	(4.36)	(4.56)	(4.74)
	[4.17]	[4.38]	[4.56]	[4.73]
3	4.537	4.72	4.89	5.05
	(4.537)	(4.72)	(4.89)	(5.06)
	[4.54]	[4.72]	[4.9]	[5.05]
4	4.88	5.04	5.195	5.35
	(4.9)	(5.04)	(5.198)	(5.35)
	[4.89]	[5.04]	[5.195]	[5.35]

Table 4: Dirac results for part of the J/Ψ mass spectra (in GeV) with $S(r) = Ar$, $A = 0.137 \text{ GeV}^2$, $m=1.12 \text{ GeV}$, and $j = l + 1/2$. The values in round parentheses are those of SLNT [12]. The values in square parentheses are those of Ref.[17].

n_r	l			
	0	1	2	3
0	3.094	3.432	3.712	3.958
	(3.106)	(3.449)	(3.733)	(3.982)
	[3.103]	[3.442]	[3.725]	[3.973]
1	3.691	3.935	4.158	4.363
	(3.698)	(3.948)	(4.175)	(4.383)
	[3.7]	[3.946]	[4.170]	[4.377]
2	4.146	4.348	4.538	4.718
	(4.152)	(4.359)	(4.553)	(4.736)
	[4.158]	[4.36]	[4.551]	[4.732]
3	4.531	4.707	4.876	5.037
	(4.535)	(4.717)	(4.889)	(5.054)
	[4.545]	[4.72]	[4.89]	[5.053]
4	4.870	5.03	5.183	5.331
	(4.874)	(5.038)	(5.195)	(5.345)
	[4.886]	[5.043]	[5.198]	[5.346]

Table 5: Dirac results for part of J/Ψ mass spectra (in GeV) with $S(r) = Ar$, $A = 0.137\text{GeV}^2$ and $m = 1.12\text{GeV}$, and $j = l - 1/2$. The values in round parentheses are those of SLNT [12]. The values in square parentheses are those of Ref.[17].

n_r	l			
	1	2	3	4
0	3.4726	3.7611	4.0118	4.2369
	(3.471)	(3.760)	(4.010)	(4.236)
	[3.47]	[3.757]	[4.006]	[4.23]
1	3.9646	4.1961	4.4069	4.6020
	(3.954)	(4.194)	(4.406)	(4.600)
	[3.965]	[4.194]	[4.403]	[4.597]
2	4.3713	4.5699	4.7552	4.9295
	(4.370)	(4.568)	(4.754)	(4.920)
	[4.374]	[4.560]	[4.753]	[4.926]
3	4.7263	4.9030	5.0700	5.2290
	(4.726)	(4.902)	(5.069)	(5.228)
	[4.731]	[4.95]	[5.07]	[\dots]
4	5.0456	5.2062	5.3595	5.5065
	(5.045)	(5.205)	(5.359)	(5.507)
	[5.053]	[5.21]	[5.361]	[5.506]

Table 6: Spin- averaged Dirac eigenvalues $\epsilon_{n_r l}$, Eq(30), for the equally mixed scalar and 4-vector power-law potential $V(r) = S(r) = Ar^\nu + V_o$, $V_o = -2.028 GeV$, $\nu = 0.1$, $A = a^{\nu+1}$, $a = 1.709$, and $m_c = 1.6179 GeV$.

$c\bar{c}$				
n_r	l	Ref.[19]	SLNT [5]	SLET
0	0	1.2364	1.2400	1.2358
1	0	1.3347	1.3400	1.3343
2	0	1.3923	1.3980	1.3915
3	0	1.4335	1.4390	1.4326
4	0	1.4657	1.4710	1.4648
0	1	1.3071	1.3090	1.3072
1	1	1.3731	1.4110	1.3730
0	2	1.3544	1.3580	1.3540

Table 7: Spin-averaged Dirac mass spectra (in Gev units), $M_{n_r l} = 2E$, for the $b\bar{b}$ system in the equally mixed scalar and 4-vector power-law potential of Eq.(29).

n_r	l	Ref.[19]	SLET
0	0	9.4390	9.4347
1	0	10.0260	10.0228
2	0	10.3694	10.3640
3	0	10.6147	10.6084
4	0	10.8063	10.800
0	1	9.8613	9.8612
1	1	10.2549	10.2537
0	2	10.1435	10.1406

Table 8: Spin-averaged Dirac mass spectra (in Gev units), $M_{n_r l} = 2E$, for the $s\bar{s}$ system in the equally mixed scalar and 4-vector power-law potential of Eq.(29).

n_r	l	Ref.[19]	SLET
0	0	1.0040	0.9999
1	0	1.5532	1.5500
2	0	1.8804	1.8752
3	0	2.1162	2.1099
4	0	2.3012	2.2950
0	1	1.3976	1.3974
1	1	1.7710	1.7697
0	2	1.6647	1.6619

Table 9: Spin-averaged Dirac mass spectra (in Gev units), $M_{n_r l} = E_d + m_c$, for the $c\bar{d}$ system in the equally mixed scalar and 4-vector power-law potential of Eq.(29).

n_r	l	Ref [19]	SLET
0	0	1.9730	1.9713
1	0	2.2107	2.2093
2	0	2.3601	2.3577
3	0	2.4699	2.4671
4	0	2.5571	2.5542
0	1	2.1413	2.1413
1	1	2.3097	2.3092
0	2	2.2612	2.2599